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BIASED SAMPLING;
THE NONCENTRAL HYPERGEOMETRIC PROBABILITY DISTRIBUTION

BY
KENNETH TED WALLENIUS

TECHNICAL REPORT NO. 70
NOVEMBER 29, 1963

SUPPORTED BY THE ARMY, NAVY AND AIR FORCE UNDER
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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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0. <u>Preliminaries</u>: Suppose a lot of transistors is to be divided among two purchasers. The lot is dichotomized by some quality criterion and let us assume there are m high quality items and n other items of tolerable quality. Purchaser G, who has contracted for r items, suspects the supplier of favoring purchaser H with respect to the m high quality items. He therefore decides to test the hypothesis that the lot has been divided in a random manner against the alternative that the supplier is biased in favor of purchaser H. Fisher's exact test is appropriate to test the null hypothesis but in order to examine the sensitivity of the test one must define and parametrize "degree of bias" and obtain the non null distribution.

This report gives an intuitively appealing meaning to "degree of bias" when sampling from a finite population and obtains a distribution herein called the noncentral hypergeometric probability distribution. Various properties of the distribution are studied and asymptotic results given.

SECTION 1

INTRODUCTION

1.1. <u>The problem</u>: Consider a set S of ℓ elements which is dichotomized in some manner (say, by an observable characteristic) into subsets M and N containing m and $n = \ell - m$ elements, respectively, and a sampling mechanism which in some way selects a subset R of r elements from S. Let a be a realization of the random variable A denoting the number of elements in $R \cap M$. The results of such an experiment are usually displayed in the form of a 2×2 contingency table such as

Table 1.1

a	c	m
b	d	n
r	s	ℓ

where
$$b = r - a$$
$$c = m - a$$
$$d = n - r + a$$
$$s = \ell - r$$

We wish to test the hypothesis that the sampling mechanism is random against the alternative of "non-randomness". The notion of random sampling from a finite population is well defined by

<u>Definition 1.1</u>: A sample, each of whose elements is drawn from a finite population, is said to be a random sample if, at each draw, all elements available for selection have an equal probability of being selected [8].

Under the hypothesis of randomness, the random variable A has a hypergeometric distribution with parameters m,n and r and hence Fisher's "exact" test [6] is applicable, i.e. the hypothesis of randomness is accepted if and only if $L \leq a \leq U$ where L and U are percentage points of the appropriate hypergeometric distribution and depend on the specified significance level α. The alternative of "non-random selection", however, is not well defined and hence discussions of the sensitivity of the test are impossible.

The purpose of this paper is to define non-random or biased selection in an intuitively appealing way and to parametrize the degree of bias in order to obtain the non-null distribution of the test statistic A and thereby determine operating characteristics of Fisher's "exact" test.

In order to relate the model under consideration to other models which yield data in the form of table 1.1, a brief survey of statistical inference based on 2×2 tables is given below.

1.2. 2 × 2 tables: Several types of statistical investigations lead to data displayable in the form of table 1.1. The particular "table" obtained as the result of a random experiment can be compared to the set of all realizable tables and thus provide a basis for inference. G. A. Barnard [1] and E. S. Pearson [9] have suggested classifying models by the number of fixed marginal totals. For example, if m, n, and r are fixed then specifying a value $A = a$ completely determines the other entries so that, in this case, the sample space is one dimensional. If r is not fixed, then the sample space is two dimensional since one needs to specify a value $(A,B) = (a,b)$ to completely determine the table. These classifications, together with the names used by Barnard and

Pearson are given below in

Table 1.2

Fixed Parameters	Classification		Sample Space
	Barnard	Pearson	
m,n,r	Independence trial	I	$\begin{array}{ccc} A & & m \\ & & n \\ r & & \end{array} \longleftrightarrow A$ $\max(0, r-n) \le A \le \min(r,m)$
m,n	Comparative trial	II	$\begin{array}{ccc} A & & m \\ B & & n \\ & & \end{array} \longleftrightarrow (A,B)$ $0 \le A + B \le m + n, A \le m$
ℓ	Double Dichotomy	III	$\begin{array}{ccc} A & C & \\ B & & \\ & & \ell \end{array} \longleftrightarrow (A,B,C)$ $0 \le A + B + C \le \ell$

Classically, the type II table is pertinent to the problem of comparing binomial parameters of 2 large populations from which random samples of sizes m and n have been drawn while the type III table is used in connection with testing a single large population for independence of 2 characteristics based on a sample of size ℓ. In these two cases, uniformly most powerful unbiased tests exist [7] and the sensitivity of such tests are discussed, for example, in [2], [4], [11], [12] and [14]. Type I tables are frequently employed in connection with testing the hypothesis that two treatments, say T_1 and T_2, are identical in so far as producing a "reaction" in a group of ℓ individuals which has been randomly divided into two groups so that m receive T_1 and $n = \ell - m$ receive T_2. Without further assumptions, the inferences pertain only to the reactions of the group being tested. Indeed, it may be impossible to repeat the experiment if, for example, reaction implies death or some other less abrupt change in susceptibility. Pearson [10] considers a particular one-sided sensitivity analysis which, however, is not applicable to the problem under consideration.

SECTION 2

BIASED SAMPLING

2.1. <u>Definition and Parametrization</u>: Imagine a sampling mechanism sequentially drawing a sample of size r from the set described in section 1.1. Prior to the k^{th} draw, $k=1,\ldots,r$, let $m(k)$ and $n(k)$ denote the number of elements remaining in sets M and N, respectively, and let

$$X(k) = \begin{cases} 1 \text{ if } k^{th} \text{ draw is an element of } M \\ 0 \text{ otherwise} \end{cases}$$

$$A = \sum_{k=1}^{r} X(k) .$$

<u>Definition 2.1</u>: The sampling mechanism is biased by θ, $0 < \theta < \infty$, if and only if

$$\frac{P\{X(k) = 1\}}{P\{X(k) = 0\}} = \frac{m(k)}{\theta n(k)} . \tag{2.1.1}$$

<u>Remarks</u>:

(1) Replacing $P\{X(k) = 0\}$ by $1-P\{X(k) = 1\}$ we obtain

$$P\{X(k) = 1\} = \frac{m(k)}{m(k) + \theta n(k)} . \tag{2.1.2}$$

(2) If $m(k) = n(k)$ then $P\{X(k) = 0\} = \theta\, P\{X(k) = 1\}$ so that definition 2.1 has the intuitively desirable property that when the remaining sets are equal in size, a bias of θ implies the sampling mechanism is "θ times more likely" to select an element from N than from M.

(3) If, at the k^{th} draw, $m(k) = \theta\, n(k)$ then $P\{X(k) = 1\} = P\{X(k) = 0\} = 1/2$. Hence, even though the sampling mechanism is biased by θ, it is equally likely that the k^{th} draw will yield an element from M as an element from N when the elements remaining in M are θ times more numerous than those remaining in N.

(4) $\lim\limits_{\theta \to \infty} P\{X(k) = 1\} = \begin{cases} 0 & \text{if } m(k) = 0 \\ 1 & \text{if } m(k) \neq 0 \end{cases}$ Hence, for a large degree of bias, the k^{th} draw will be an element from set M provided M has not been totally depleted.

(5) For $\theta = 1$, definitions 2.1 and 1.1 coincide and clearly A has a hypergeometric probability distribution with parameters m, n, and r. For this reason, the distribution of A for $0 < \theta < \infty$ will be called noncentral hypergeometric with noncentrality parameter θ and will be denoted $P_\theta\{A = a\}$.

2.2 <u>Example</u>: The main result of this paper was motivated by the following genetics study [3]. A small rocky island located about 30 miles seaward of San Francisco Bay has supported a population of rabbits for at least 100 years. This population affords the geneticist an ideal nonlaboratory environment in which to study genetic fixation 1/ because

1/ Genetic fixation in a specified population is the tendency for all but one allele to disappear from a particular gene locus. [13]

(a) Migration of animals in nonexistent

(b) The population is greatly reduced each summer by the sparce supply of food [2]

(c) Absence of predators

(d) The terrian facilitates trapping to the extent that it is safe to assume that sampling error is negligible.

One of the loci under consideration is blood type. If no genetic selection[3] has taken place in the population, the number of generations and the effective size of the breeding population can be used to determine the probability of fixation [5]. For the case in question, this probability is approximately 0.04. Since the blood type locus was observed to be polymorphic,[4] a study was made to determine if selection was occuring in neonate survival. Neonates were captured, blood typed, released and, after three months, the surviving animals were recaptured. The data are given below in

Table 2.2

	Survived	Died	Totals
Homozygotes	41	54	95
Heterozygotes	34	86	120
Totals	75	140	215

2/ In 1961, for example, only 69 of 1200 neonate rabbits survived to an age where reproduction could occur.

3/ Changes in allele frequence between generations, other than those attributable to random gamete and survival sampling, are called selective changes. [13]

4/ More than one allele present in the population at the locus in question; the opposite of genetic fixation. [13]

The hypothesis to be tested is that the sample of size 75 was drawn randomly by "nature" from the population of size 215 against the alternative that nature is biased (i.e., selection against one phenotype or the other by the mechanism of death had occured). To test this hypothesis, one simply applies Fisher's exact test at an appropriate level using the values

$$\begin{cases} m = 95 \\ n = 120 \\ r = 75 \\ A = 41 \end{cases}.$$

Attempting to parametrize the alternative and obtain the power function of the test motivated this paper. After the noncentral hypergeometric distribution is obtained in section 3, it will be applied in section 5 to a discussion of the power of the above test.

SECTION 3

THE NONCENTRAL HYPERGEOMETRIC DISTRIBUTION

3.1. A First Expression: In this subsection we shall obtain an expression for $P_\theta\{A = a\}$ by a straight forward enumeration method. This form, although not mathematically appealing, will be used in a discussion of bounds on $P_\theta\{A = a\}$ and in certain asymptotic results in section 4. A simpler and more desirable expression will be obtained in subsection 3.2.

Using the notation introduced in Section 2, let $X = \{X(1), X(2), \ldots, X(r)\}$ be the random variable denoting the outcome of drawing r elements from $M \cup N$ and $x = \{x(1), x(2), \ldots, x(r)\}$ be a realization on X. Then, using (2.1.2) and $P\{X = x\} = P\{X(1) = x(1)\} \times$ $P\{X(2) = x(2) \mid X(1)\}\ P\{X(3) = X(3) \mid X(1), X(2)\} \cdots\cdots$ $P\{X(r) = x(r) \mid X(1), X(2), \ldots X(r-1)\}$ we obtain

$$P\{X = x\} = \begin{cases} \displaystyle\prod_{k=1}^{r} \frac{x(k)\ m(k,x) + [1-x(k)]\ \theta\ n(k,x)}{m(k,x) + \theta\ n(k,x)} & \\ \qquad\qquad \max\{o, r-n\} \le \displaystyle\sum_{k=1}^{r} x(k) \le \min\{r,m\} & \\ 0 \qquad\qquad \text{otherwise} & \end{cases} \tag{3.1.1}$$

where $m(k,x)$ and $n(k,x)$ are the number of elements remaining in M and N respectively prior to the k^{th} draw. Actually, $m(k,x)$ and $n(k,x)$ depend only on $x(1), x(2), \ldots, x(k-1)$; more precisely,

$$\begin{aligned} m(k,x) &= m - \sum_{j=1}^{k-1} x(j) \\ n(k,x) &= n - \sum_{j=1}^{k-1} [1-x(j)]\,. \end{aligned} \tag{3.1.2}$$

Then, $P_\theta\{A = a\} = \sum_{x \in X(a)} P\{X = x\}$ where $X(a) = \{x \mid \sum_{k=1}^{r} x(k) = a\}$ (3.1.3)

Viewing the $\binom{r}{a}$ elements in $X(a)$ as binary numbers instead of vectors for a moment, let us order $X(a)$ by decreasing magnitude and index the ordered set with j.

i.e. $j = 1$ $\{1, 1, \ldots 1, 1, 0, 0, \ldots, 0\}$

$j = 2$ $\{1, 1, \ldots 1, 0\ 1, 0, \ldots, 0\}$

$\vdots$

$j = \binom{r}{a}$ $\{0, 0, \ldots 0, 1, 1, \ldots, 1\}$

where there are exactly a 1's and r-a 0's. Denote the j^{th} x in the ordering of $X(a)$ by x_j. By using (3.1.1) and (3.1.3) and observing that for each $x_j \in X(a)$,

$$\prod_{k=1}^{r} \left\{x_j(k)\ m((k,x_j) + [1 - x_j]\ \theta\ n\ (k, x_j)\right\} = (m)_a(n)_{r-a}\theta^{r-a} \quad \text{where}$$

$(m)_a = m(m-1) \ldots. (m-a+1)$ we obtain

$$P_\theta\{A=a\} = (m)_a(n)_{r-a}\theta^{r-a} \sum_{j=1}^{\binom{r}{a}} \prod_{k=1}^{r} \left[m(k,x_j) + \theta n\ (k,x_j)\right]^{-1}. \tag{3.1.4}$$

As a verification, note that for $\theta = 1$

$$\prod_{k=1}^{r} \left[m(k,x_j) + \theta n\ (k,x,)\right]^{-1} = \left[(\ell)_r\right]^{-1}$$

so that $P\{A=a\} = \dfrac{(m)_a\ (n)_{r-a}}{(\ell)_r}$ which is the central hypergeometric case.

3.2 A Preferred Expression: While (3.1.4) is correct, it is a complicated notational expression and leaves much to be desired. The attack used in this subsection is that of partial difference equations. In order to write down a partial difference equation we shall need to modify the notation for $P_\theta(A=a)$ in order to include reference to the particular values of the parameters. Therefore, let us replace

$$P_\theta\{A=a\} \text{ by } P_\theta\{a;\ m,\ n,\ r\}.$$

Conditioning on $X(1)$ we obtain

$$\begin{aligned}
P_\theta\{a;\ m,\ n,\ r\} &= E\left[P_\theta\{a;\ m,\ n,\ r \,\middle|\, X(1)\}\right] \\
&= P_\theta\{a;\ m,\ n,\ r \,|\, X(1) = 1\}\, P\{X(1) = 1\} \\
&\quad + P_\theta\{a;\ m,\ n,\ r \,|\, X(1) = 0\}\, P\{X(1) = 0\} \\
&= P_\theta\{a-1;\ m-1,\ n,\ r-1\}\frac{m}{m+n\theta} + P_\theta\{a;\ m,\ n-1,\ r-1\}\frac{n\theta}{m+n\theta}\ .
\end{aligned}$$

Multiplying by $m + n\theta$ we obtain

$$(m+n\theta)\ P_\theta\{a;\ m,\ n,\ r\} = m\ P_\theta\{a-1;\ m-1,\ n,\ r-1\} + n\theta\ P_\theta\{a;\ m,\ n-1,\ r-1\} \tag{3.2.1}$$

with boundary conditions

$$P_\theta\{a;\ 0,\ n,\ r\} = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \end{cases} \tag{3.2.2}$$

$$P_\theta\{a;\ m,\ 0,\ r\} = \begin{cases} 0 & a < r \\ 1 & a = r \end{cases} \tag{3.2.3}$$

$$P_\theta\{a;\ m,\ n,\ 0\} = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \end{cases} \tag{3.2.4}$$

$$P_\theta\{0;\ m,\ n,\ r\} = \frac{(n)_r}{(\frac{m}{\theta} + n)_r}\ . \tag{3.2.5}$$

The first three boundary conditions need no explanation; the fourth is obtained directly from (3.1.4).

Now (3.2.1) is a partial difference equation in four variables. Often, by assuming the discrete variables are continuous and viewing the partial difference equation as a partial differential equation, a simplification of the former will be suggested by the form of the solution of the latter. Rewriting (3.2.1) as

$m(P_\theta\{a;\ m,\ n,\ r\} - P_\theta\{a-1;\ m-1,\ n,\ r-1\}) + n\theta\ (P_\theta\{a;\ m,\ n,\ r\} -$

$P_\theta\{a;\ m,\ n-1\ r-1\}) = 0$ and, by adding and subtracting appropriate terms we obtain

$$\begin{aligned}
&m(P_\theta\{a;\ m,\ n,\ r\} - P_\theta\{a;\ m-1,\ n,\ r\}) \\
&+ m(P_\theta\{a;\ m-1,\ n,\ r\} - P_\theta\{a;\ m-1,\ n,\ r-1\}) \\
&+ m(P_\theta\{a;\ m-1,\ n,\ r-1\} - P_\theta\{a-1;\ m-1,\ n,\ r-1\}) \\
&+ n\theta\ (P\{a;\ m,\ n,\ r\} - P_\theta\{a;\ m,\ n-1,\ r\}) \\
&+ n\theta(P\{a;\ m,\ n-1,\ r\} - P\{a;\ m,\ n-1,\ r-1\}) = 0
\end{aligned} \tag{3.2.6}$$

which leads to

$$m\left(\frac{\partial P}{\partial m} + \frac{\partial P}{\partial r} + \frac{\partial P}{\partial a}\right) + \theta n\left(\frac{\partial P}{\partial n} + \frac{\partial P}{\partial r}\right) = 0. \tag{3.2.7}$$

Applying the method of characteristics,

$$\frac{dm}{m} = \frac{dr}{m+n\theta} = \frac{da}{m} = \frac{dn}{n\theta} = \frac{dp}{0}\ . \tag{3.2.8}$$

The first and third members of (3.2.8) yield

$$m = a + c_1\ . \tag{3.2.9}$$

The first and fourth members of (3.2.8) yield

$$\ln n = \theta \ln m + \ln C_2 \tag{3.2.10}$$

$$\text{or} \quad n = C_2 m^{\theta} .$$

The first and second members of (3.2.8) yield

$$\frac{dr}{dm} = \frac{m+n\theta}{m}$$

$$= 1 + \frac{n\theta}{m}$$

$$= 1 + \theta C_2 m^{\theta-1} \text{ which has the solution:}$$

$$r = m + C_2 m^{\theta} + C_3$$

$$= m + n + C_3 . \tag{3.2.11}$$

Hence, the solution of (3.2.7) will be some function of (c_1, c_2, c_3) or, using (3.2.9), (3.2.10) and (3.2.11)

$$P_{\theta}\{a; m, n, r\} = F(m-a, nm^{-\theta}, r-m-n) . \tag{3.2.12}$$

This result suggests the following change of variables. Let

$$P_{\theta}\{a; m, n, r\} = Q(m+n-r, m-a, m, n). \tag{3.2.13}$$

Then (3.2.1) becomes

$$(m+n\theta)\, Q\, (m+n-r, m-a, m, n) = m\, Q\, (m+n-r, m-a, m-1, n)$$

$$+ n\theta Q\, (m+n-r, m-a, m, n-1) . \tag{3.2.14}$$

Notice that the first two arguments, (m+n-r) and (m-a) are identical in each term of (3.2.14) so that the number of variables has been reduced from four to two. In terms of Q, the boundary conditions

(3.2.2) through (3.2.5) become

$$Q(n-r,\ -a,\ 0,\ n) = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \end{cases} \tag{3.2.15}$$

$$Q(m-r,\ m-a,\ m,\ 0) = \begin{cases} 0 & a < r \\ 1 & a = r \end{cases} \tag{3.2.16}$$

$$Q(m+n,\ m-a,\ m,\ n) = \begin{cases} 0 & a > 0 \\ 1 & a = 0 \end{cases} \tag{3.2.17}$$

$$Q(m+n-r,\ m,\ m,\ n) = \frac{(n)_r}{(\frac{m}{\theta} + n)_r} . \tag{3.2.18}$$

Now, define

$$\Phi_{m,n}(x,y) = \sum_{r=0}^{m+n-1} \sum_{a=\max\{o,r-n\}}^{\min\{r,m\}} Q(m+n-r,\ m-a,\ m,n)x^r\ y^a \tag{3.2.19}$$

Boundary conditions on $\Phi_{m,n}(x,y)$ are easily obtained from (3.2.15) and (3.2.16).

$$\Phi_{0,n}(x,y) = \sum_{r=0}^{n-1} Q(n-r,\ 0,\ 0,\ n)\ x^r = \frac{x^n-1}{x\ -1} \tag{3.2.20}$$

$$\Phi_{m,0}(x,y) = \sum_{r=0}^{m-1} Q(m-r,\ m-r,\ m,0)\ x^r\ y^r = \frac{(xy)^m-1}{xy-1} \tag{3.2.21}$$

Multiplying (3.2.19) by $m+n\theta$ and splitting off the term for $r=0$ we obtain

$$(m+n\theta)\ \Phi_{m,n}(x,y) = \sum_{r=1}^{m+n-1} \sum_{a=\max\{o,r-n\}}^{\min\{r,m\}} (m+n\theta)\ Q\ (m+n-r,m-a,m,n)x^r y^a + (m+n\theta)$$

and, applying (3.2.14) to the right hand side, we obtain

$$(m+n\theta)\ \Phi_{m,n}(x,y) = m \sum_{r=1}^{m+n-1} \sum_{a=\max\{o,r-n\}}^{\min\{r,m\}} Q(m+n-r,\ m-a,\ m-1,\ n)x^r\ y^a$$

$$+\ n\theta \sum_{r=1}^{m+n-1} \sum_{a=\max\{o,r-n\}}^{\min\{r,m\}} Q(m+n-r,\ m-a,\ m,\ n-1)x^r\ y^a$$

$$+\ (m + n\theta) . \tag{3.2.22}$$

In the first term of the right hand side of (3.2.22), $\max\{o, r-n\}$ can be replaced by $\max\{1, r-n\}$ since $Q(m+n-r, m, m-1,n) = 0$. Then, replacing a by $a+1$, we obtain

$$m \sum_{r=1}^{m+n-1} \sum_{a=\max\{0,r-1-n\}}^{\min\{r-1,m-1\}} Q(m-1+n-[r-1], m-1-a, m-1,n)x^r y^{a+1}$$

and, replacing r by $r+1$, we have

$$m \sum_{r=0}^{m-1+n-1} \sum_{a=\max\{0,r-n\}}^{\min\{r,m-1\}} Q(m-1+n-r,m-1-a,m-1,n)x^{r+1} y^{a+1} = mxy\Phi_{m-1,n}(x,y). \tag{3.2.23}$$

In the second term of the right hand side of (3.2.22), $\min\{r,m\}$ can be replaced by $\min\{r-1, m\}$ since $Q(m+n-r, m-r, m, n-1) = 0$. Then, replacing r by $r+1$ we obtain

$$n\theta \sum_{r=0}^{m+n-1-1} \sum_{a=\max\{0,r-[n-1]\}}^{\min\{r,m\}} Q(m+n-1-r,m-a,m,n-1)x^{r+1} y^a = n\theta x\Phi_{m,n-1}(x,y). \tag{3.2.24}$$

Substituting (3.2.23) and (3.2.24) into (3.2.22) yields

$$(m+n\theta)\, \Phi_{m,n}(x,y) = mxy\, \Phi_{m-1,n}(x,y) + n\,\theta\, x\, \Phi_{m,n-1}(x,y) + (m+n\theta)\,. \tag{3.2.25}$$

Let $\psi(\xi,\eta)$ be the generating function of $\Phi_{m,n}(x,y)$:

$$\psi(\xi,\eta) = \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \Phi_{m,n} \frac{\xi^m \eta^n}{m!\; n!}\,. \tag{3.2.26}$$

Boundary conditions on $\psi(\xi,\eta)$ are easily obtained from (3.2.20) and (3.2.21) as

$$\psi(0,\eta) = \sum_{n=0}^{\infty} \Phi_{0,n} \frac{\eta^n}{n!}$$

$$= \frac{1}{x-1} \sum_{n=0}^{\infty} \frac{(x^n-1)\eta^n}{n!}$$

$$= \frac{e^{x\eta} - e^{\eta}}{x-1} \qquad (3.2.27)$$

$$\psi(\xi,0) = \sum_{m=0}^{\infty} \Phi_{m,0} \frac{\xi^m}{m!}$$

$$= \frac{1}{xy-1} \sum_{m=0}^{\infty} \left[(xy)^m - 1\right] \xi^m$$

$$= \frac{e^{xy\xi} - e^{\xi}}{xy-1} \quad . \qquad (3.2.28)$$

From (3.2.26) one obtains by differentiation:

$$\xi \ \psi_\xi(\xi,\eta) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} m \ \Phi_{m,n} \frac{\xi^m \eta^n}{m!n!}$$

$$\eta\theta \ \psi_\eta(\xi,\eta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} n\theta \ \Phi_{m,n} \frac{\xi^m \eta^n}{m!n!} \quad \text{and hence}$$

$$\xi \ \psi_\xi(\xi,\eta) + \eta \ \theta \ \psi_\eta(\xi,\eta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m+n\theta) \ \Phi_{m,n} \frac{\xi^m \eta^n}{m!n!}$$

$$+ \sum_{m=1}^{\infty} m \ \Phi_{m,0} \frac{\xi^m}{m!} + \sum_{n=1}^{\infty} n\theta \ \Phi_{0,n} \frac{\eta^n}{n!} \quad . \qquad (3.2.29)$$

Substituting (3.2.25) into the first term on the right hand side of (3.2.29) yields

$$xy \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m\Phi_{m-1,n} \frac{\xi^m \eta^n}{m!n!} + x\theta \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n\Phi_{m,n-1} \frac{\xi^m \eta^n}{m!n!} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m \frac{\xi^m \eta^n}{m!n!}$$

$$+ \theta \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n \frac{\xi^m \eta^n}{m!n!} \qquad (3.2.30)$$

Taking the terms of (3.2.30) one at a time:

$$xy \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m\, \Phi_{m-1,n} \frac{\xi^m \eta^n}{m!n!}$$

$$= xy \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Phi_{m,n} \frac{\xi^{m+1} \eta^n}{m!\ n!}$$

$$= \xi xy \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi_{m,n} \frac{\xi^m \eta^n}{m!n!} - xy\xi\psi(\xi,0)$$

$$= \xi xy\psi(\xi,\eta) - xy\xi\left(\frac{e^{xy\xi} - e^{\xi}}{xy - 1}\right) \qquad (3.2.31)$$

$$x\theta \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n\Phi_{m,n-1} \frac{\xi^m \eta^n}{m!n!}$$

$$= x\theta \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \Phi_{m,n} \frac{\xi^m \eta^{n+1}}{m!n!}$$

$$= x\theta\eta \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Phi_{m,n} \frac{\xi^m \eta^n}{m!n!} - x\theta\eta\psi(0,\eta)$$

$$= x\theta\eta\psi(\xi,\eta) - x\theta\eta\left(\frac{e^{x\eta} - e^{\eta}}{x - 1}\right) \qquad (3.2.32)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m \frac{\xi^m \eta^n}{m!n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\xi^{m+1}}{m!} \frac{\eta^n}{n!}$$

$$= \xi\left(e^{\xi+\eta} - e^{\xi}\right) \qquad (3.2.33)$$

$$\theta \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n \frac{\xi^m \eta^n}{m!n!}$$

$$= \theta \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\xi^m \eta^{n+1}}{m!n!}$$

$$= \eta\theta \left(e^{\xi+\eta} - e^{\eta} \right) . \tag{3.2.34}$$

The second term on the right hand side of (3.2.29), after applying (3.2.21) becomes

$$\sum_{m=1}^{\infty} m \left(\frac{(xy)^m - 1}{xy-1} \right) \frac{\xi^m}{m!}$$

$$= \frac{1}{xy-1} \left(\sum_{m=1}^{\infty} \frac{(xy\xi)^m}{(m-1)!} - \sum_{m=1}^{\infty} \frac{\xi^m}{(m-1)!} \right)$$

$$= \frac{1}{xy-1} \left(xy\xi e^{xy\xi} - \xi e^{\xi} \right) . \tag{3.2.35}$$

Similarly, the third term on the right hand side of (3.2.29), after applying (3.2.20), becomes

$$\frac{\theta}{x-1} \left(x\eta e^{x\eta} - \eta e^{\eta} \right) . \tag{3.2.36}$$

Substituting (3.2.31) through (3.2.36) into (3.2.29) and simplifying the resulting expression yields:

$$\xi \left[\psi_{\xi}(\xi,\eta) - xy\psi(\xi,\eta) \right] + \theta\eta \left[\psi_{\eta}(\xi,\eta) - x\psi(\xi,\eta) \right] = (\xi + \theta\eta) e^{\xi+\eta} \tag{3.2.37}$$

We now seek a solution of (3.2.37) which satisfies the boundary conditions (3.2.27) and (3.2.28).

Let $\mathcal{U}(\xi,\eta) = e^{-xy\xi - x\eta}\,\psi(\xi,\eta)$.

Then $\mathcal{U}_{\xi}(\xi,\eta) = e^{-xy\xi - x\eta}\left[\psi_{\xi}(\xi,\eta) - xy\psi(\xi,\eta)\right]$

$$\mathcal{U}_{\eta}(\xi,\eta) = e^{-xy\xi - x\eta}\left[\psi_{\eta}(\xi,\eta) - x\psi(\xi,\eta)\right]$$

and (3.2.37) becomes

$$\xi\mathcal{U}_{\xi}(\xi,\eta) + \theta\eta\mathcal{U}_{\eta}(\xi,\eta) = (\xi + \theta\eta)e^{-\xi(xy-1)-\eta(x-1)} . \qquad (3.2.38)$$

The boundary conditions become

$$\mathcal{U}(0,\eta) = \frac{1-e^{-\eta(x-1)}}{x-1} \qquad (3.2.39)$$

$$\mathcal{U}(\xi,0) = \frac{1-e^{-\xi(xy-1)}}{xy-1} . \qquad (3.2.40)$$

The Mellin transform of a function $\mathcal{U}(\xi,\eta)$ with respect to η (ξ will be treated as a parameter) is defined to be

$$\mathfrak{U}(\xi,s) = \int_0^\infty \eta^{s-1}\mathcal{U}(\xi,\eta)d\eta \quad \text{where}$$

s is a complex variable suitably restricted to insure convergence, if the transform exists at all. Taking the Mellin transform of both sides of (3.2.38) yields

$$\xi \int_0^\infty \eta^{s-1} U_\xi(\xi,\eta)d\eta + \theta \int_0^\infty \eta^{s-1} \left[\eta U_\eta(\xi,\eta)\right] d\eta$$

$$= e^{-\xi(xy-1)} \left[\xi \int_0^\infty \eta^{s-1} e^{-(x-1)\eta} d\eta + \theta \int_0^\infty \eta^{s} e^{-(x-1)\eta} d\eta\right].$$

Simplifying, we obtain

$$\xi \mathcal{U}_\xi(\xi,s) - \theta s \mathcal{U}(\xi,s) = e^{-\xi(xy-1)} \left[\xi(x-1)^{-s}\Gamma(s) + \theta(x-1)^{-(s+1)}\Gamma(s+1)\right]$$

and, multiplying by the integrating factor $\xi^{-\theta s-1}$ gives

$$\frac{\partial}{\partial \xi}\left[\xi^{-\theta s}\mathcal{U}(\xi,s)\right] = \xi^{-\theta s-1} e^{-\xi(xy-1)} \left[\xi(x-1)^{-s}\Gamma(s) + \theta(x-1)^{-(s+1)}\Gamma(s+1)\right]$$

and hence,

$$\mathcal{U}(\xi,s) = \xi^{\theta s} \int_C^\xi \tau^{-\theta s-1} e^{-\tau(xy-1)} \left[\tau(x-1)^{-s}\Gamma(s)+\theta(x-1)^{-(s+1)}\Gamma(s+1)\right] d\tau$$

where C is a constant of integration which may be a function of η. Since the mathematics is simpler and a solution satisfying the boundary conditions is obtained by taking $C=0$, let us do so; also, let us make the change of variable $t = \frac{\tau}{\xi}$.

$$\mathcal{U}(\xi,s) = \int_0^1 t^{-\theta s-1} e^{-\xi(xy-1)t}\left[\xi t(x-1)^{-s}\Gamma(s) + \theta(x-1)^{-(s+1)}\Gamma(s+1)\right] dt.$$

Inverting, we obtain

$$U(\xi,\eta) = \int_0^1 e^{-\left[\xi(xy-1)t + \eta(x-1)t^\theta\right]} \left(\xi + \theta\eta t^{\theta-1}\right) dt. \tag{3.2.41}$$

$$\mathcal{U}(0,\eta) = \int_0^1 e^{-\eta(x-1)t^\theta} \theta\eta t^{\theta-1} dt$$

$$= \frac{1}{x-1} \int_0^1 e^{-\eta(x-1)t^\theta} \eta\theta(x-1)t^{\theta-1} dt$$

$$= \frac{1}{x-1} e^{-\eta(x-1)t^\theta} \Big|_1^0$$

$$= \frac{1}{x-1} (1-e^{-\eta(x-1)}) \text{ which checks with (3.2.39)}$$

$$\mathcal{U}(\xi,0) = \int_0^1 e^{-\xi(xy-1)t} \xi \; dt$$

$$= \frac{1}{xy-1} \; e^{-\xi(xy-1)t} \Big|_1^0$$

$$= \frac{1}{xy-1}(1-e^{-\xi(xy-1)}) \text{ which checks with (3.2.40).}$$

Using (3.2.41) we can now recover $\psi(\xi,\eta)$

$$\psi(\xi,\eta) = \int_0^1 e^{\xi[(1-xy)t+xy] + [(1-x)t^\theta + x]}(\xi+\theta\eta t^{\theta-1})dt$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{\int_0^1 [(1-xy)t+xy]^m[(1-x)t^\theta+x]^n[\xi+\theta\eta t^{\theta-1}]dt\right\} \frac{\xi^m\eta^n}{m!n!} \quad (3.2.42)$$

From this expression, the coefficient of

$\frac{\xi^m\eta^n}{m!n!}$ is seen to be

$$\Phi_{m,n}(x,y) = m\int_0^1 \left[(1-xy)t + xy\right]^{m-1} \left[(1-x)t^\theta + x\right]^n dt$$

$$+ n\theta \int_0^1 \left[(1-xy)t + xy\right]^m \left[(1-x)t^\theta + x\right]^{n-1} t^{\theta-1} dt$$

(3.2.43)

From the definition (3.2.19) of $\Phi_{m,n}(x,y)$ we observe that

$$\sum_{a=\max\{0,r-n\}}^{\min\{r,m\}} Q(m + n-r,\ m-a,\ m,\ n)y^a = \frac{1}{r!} \left.\frac{\partial^r \Phi_{m,n}(x,y)}{\partial x^r}\right|_{x=0}$$

Considering the first term on the right hand side of (3.2.43), and applying Leibniz's rule for differentiation and interchanging the order of differentiation and intergration,

$$\frac{m}{r!} \frac{\partial^r}{\partial x^r} \int_0^1 \left[(1-xy)t + xy\right]^{m-1} \left[(1-x)t^\theta + x\right]^n dt \Big|_{x=0}$$

$$= \frac{m}{r!} \int_0^1 \left\{ \sum_{k=0}^{r} \binom{r}{k} \frac{\partial^k \left[(1-xy)t + xy\right]^{m-1}}{\partial x^k} \left.\frac{\partial^{r-k}\left[(1-x)t^\theta + x\right]^n}{\partial x^{r-k}}\right|_{x=0} \right\} dt$$

$$= \frac{m}{r!} \int_0^1 \sum_{k=0}^{r} \binom{r}{k} (m-1)_k (y[1-t])^k t^{m-1-k} (n)_{r-k} ([1-t^\theta])^{r-k} t^{\theta(n-r+k)} dt$$

$$= m \sum_{k=0}^{r} \left[\frac{(m-1)_k\ (n)_{r-k}}{k!\ r-k!} \int_0^1 (1-t)^t (1-t^\theta)^{r-k} t^{m-k-1 + \theta(n-r+k)} dt\right] y^k$$

$$= m \sum_{k=0}^{r} \left[\binom{m-1}{m} \binom{n}{r-k} \int_0^1 (1-t)^k (1-t^\theta)^{r-k} t^{m-k-1+\theta(n-r+k)} dt \right] y^k \tag{3.2.44}$$

Similarly, the second term on the right hand side of (3.2.43), when differentiated, becomes

$$n\theta \sum_{k=0}^{r} \left[\binom{m}{k} \binom{n-1}{r-k} \int_0^1 (1-t)^k (1-t^\theta)^{r-k} t^{m-k-1+\theta(n-r+k)} dt \right] y^k. \tag{3.2.45}$$

Here, it is understood that $\binom{a}{b} = 0$ if $b > a$. By combining (3.2.44) and (3.2.45) and by uniqueness of power series, we find that

$$P_\theta\{a;m,n,r\} = \left[m\binom{m-1}{a}\binom{n}{r-a} + n\theta \binom{m}{a}\binom{n-1}{r-a} \right] \times$$

$$\int_0^1 (1-t)^a (1-t^\theta)^{r-a} t^{m-a-1+\theta(n-r+a)} dt$$

Some manipulation then results in:

$$P_\theta\{a;m,n,r\} = \binom{m}{a}\binom{n}{r-a}\left[m-a+\theta(n-r+a)\right] \int_0^1 (1-t)^a (1-t^\theta)^{r-a} t^{m-a-1+\theta(n-r+a)} dt \tag{3.2.46}$$

3.3 <u>Compliments and verifications</u>: In this subsection we shall verify that (3.2.46) has required properties of a bonifide probability distribution and satisfies the defining conditions (3.2.1) through (3.2.5)

<u>Theorem 3.3.1:</u> For $r < m+n$ and $\max\{0,r-n\} \le a \le \min\{r,m\}$, $P_\theta\{a;m,n,r\}$ as given in (3.2.46) is discrete probability mass function.

<u>Proof</u>:

Clearly $P_\theta\{a; m, n, r\} \ge 0$. The generating function for

$$\sum_{a=\max\{0,r-n\}}^{\min\{r,m\}} P_\theta\{a; m, n, r\}$$

can be obtained by setting $y = 1$ in (3.2.19)

$$\Phi_{m,n}(x,1) \equiv \sum_{r=0}^{m+n-1} \left\{ \sum_{a=\max\{0,r-n\}}^{\min\{r,m\}} P_\theta(a;\, m,\, n,\, r) \right\} x^r \tag{3.3.1}$$

$$= m \int_0^1 \Big[(1-x)t + x\Big]^{m-1} \Big[(1-x)t^\theta + x\Big]^n dt$$

$$+ n\theta \int_0^1 \Big[(1-x)t + x\Big]^m \Big[(1-x)t^\theta + x\Big]^{n-1} t^{\theta-1} dt \tag{3.3.2}$$

by (3.2.43).

Integrating the first term on the right hand side of (3.3.2) by parts:

$$\text{Let } U = [(1-x)t^\theta + x]^n, \quad dV = m[(1-x)t + x]^{m-1} dt$$

$$\text{then } dU = n[(1-x)t^\theta + x]^{n-1} \theta(1-x)t^{\theta-1} dt, \; V = \frac{1}{(1-x)} \Big[(1-x)t + x\Big]^m$$

we obtain the expression

$$\frac{1}{(1-x)} \Big[(1-x)t^\theta + x\Big]^n \Big[(1-x)t + x\Big]^m \Big|_0^1 - n\theta \int_0^1 \Big[(1-x)t + x\Big]^m \Big[(1-x)t^\theta + x\Big]^{n-1} t^{\theta-1} dt$$

which, when substituted in (3.3.2) gives

$$\Phi_{m,n}(x,1) = \frac{1-x^{m+n}}{1-x} = \sum_{r=0}^{m+n-1} x^r . \tag{3.3.3}$$

Since (3.3.1) and (3.3.3) are identities in x, we have

$$\sum_{a=\max\{0,r-n\}}^{\min\{r,m\}} \binom{m}{a}\binom{n}{r-a} [m-a+\theta(n-r+a)] \int_0^1 (1-t)^a (1-t^\theta)^{r-a} t^{m-a-1+\theta(n-r+a)} dt \equiv 1 \tag{3.3.4}$$

for $r < m + n$, which completes the proof.

An integration by parts and some manipulations with the binomial coefficients if required to show that $P_\theta(a; m, n, r)$ as given by (3.2.46) satisfies the partial difference equation (3.2.1). An example of the type of computations necessary to show that the boundary conditions are satisfied is given below. The others require similar computations.

$P_\theta(a; 0, n, r)$

$$= \binom{0}{a}\binom{n}{r} [-a+\theta(n-r+a)] \int_0^1 (1-t)^a (1-t^\theta)^{r-a} t^{-a-1+\theta(n-r+a)} dt$$

$$= \begin{cases} 0 & \text{if } a > 0 \\ \binom{n}{r} [\theta(n-r)] \int_0^1 (1-t^\theta)^r t^{\theta(n-r)-1} dt & \text{if } a=0 \end{cases}$$

$$= \begin{cases} 0 & a > 0 \\ \frac{n!}{r!(n-r-1)!} \beta(r+1, n-r) & a=0 \end{cases}$$

$$= \begin{cases} 0 & a > 0 \\ 1 & a = 0 \end{cases}$$

3.4 Moments: One can always write down an expression for the k^{th} moment as a finite sum of terms simply by using the moment definition. For example, the first moment is, by definition,

$E_\theta(A; m, n, r) \equiv$

$$\sum_{a=0}^{r} a\binom{m}{a}\binom{n}{r-a} [m-a+\theta(n-r+a)] \int_0^1 (1-t)^a (1-t^\theta)^{r-a} t^{m-a-1+\theta(n-r+a)} dt. \tag{3.4.1}$$

This is not an attractive form and requires considerable computational effort to obtain a numerical result for given values of m,n,r and θ. A closed form of (3.4.1) would be desirable but unfortunately is not obtainable. However, some rather interesting side results were obtained as byproducts of attempts to simplify (3.4.1).

The first approach used was to condition $E_\theta(A;\ m,\ n,\ r)$ on the result $X(1)$ of the first draw.

$$E_\theta\{A;\ m,\ n,\ r\} = E\left[E_\theta\{A;\ m,\ n,\ r \mid X(1)\}\right]$$

$$= \left[1+E_\theta\{A;\ m-1,\ n,\ r-1\}\right] \frac{m}{m+n\theta} + E\{A;\ m,\ n-1,\ r-1\} \frac{n\theta}{m+n\theta}$$

or, simplifying,

$$(m + n\theta)\, E_\theta\{A;m,n,r\} = m\, E_\theta\{A;m-1,r-1\} + n\theta E_\theta\ \{A;m,n-1,r-1\} + m\,. \tag{3.4.2}$$

Clearly, $E_\theta\{A;\ m,\ n,\ r\}$ must satisfy the boundary conditions

$$E_\theta\{A;\ 0,\ n,\ r\} = 0 \tag{3.4.3}$$

$$E_\theta\{A;\ m,\ 0,\ r\} = r\,. \tag{3.4.4}$$

Define $D_\theta(m + n-r,\ m,\ n) = E_\theta\{A;\ m,\ n,\ r\}$

The (3.4.2) through (3.4.4) become

$$(m+n\theta)D_\theta(m+n-r,m,n) = mD_\theta(m+n-r,m-1,n) + n\theta D_\theta(m+n-r,m,n-1) + m \tag{3.4.5}$$

$$D_\theta(n-r,0,n) = 0 \tag{3.4.6}$$

$$D_\theta(m-r,m,0) = r\,. \tag{3.4.7}$$

Define $$\Omega_{m,n}(x) = \sum_{r=1}^{m+n} D_\theta(m+n-r,m,n)x^r. \tag{3.4.8}$$

$$\text{Then } \Omega_{0,n}(x) = 0 \tag{3.4.9}$$

$$\text{and } \Omega_{m,0}(x) = \sum_{r=1}^{m} rx^r = x \sum_{r=1}^{m} rx^{r-1}$$

$$= x \frac{d}{dx}\left(\frac{x^{m+1}-1}{x-1}\right)$$

$$= x \frac{mx^{m+1}-(m+1)x^m+1}{(x-1)^2} . \tag{3.4.10}$$

Multiplying (3.4.8) by $(m+n\theta)$ and using (3.4.5) gives

$$(m+n\theta)\Omega_{m,n}(x) = \sum_{r=1}^{m+n} mD_\theta(m+n-r,m-1,n)x^r + \sum_{r=1}^{m+n} n\theta D_\theta(m+n-r,m,n-1)x^r + \sum_{r=1}^{m+n} mx^r.$$

Replacing r by $r+1$ and noting the first two terms are zero when $r=0$ gives

$$(m+n\theta)\,\Omega_{m,n}(x) = mx\,\Omega_{m-1,n}(x) + n\theta x\,\Omega_{m,n-1}(x) + mx \frac{x^{m+n}-1}{x-1} . \tag{3.4.11}$$

$$\text{Now define } \Delta(\xi,\eta) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Omega_{m,n} \frac{\xi^m \eta^n}{m!n!} . \tag{3.4.12}$$

Then (3.4.9) and (3.4.10) become

$$\Delta\,(0,\eta) = 0 \tag{3.4.13}$$

$$\Delta\,(\xi,0) = \sum_{m=0}^{\infty} \Omega_{m,0}(x) \frac{\xi^m}{m!}$$

$$= \frac{x}{(x-1)^2} \sum_{m=0}^{\infty} \frac{mx^{m+1}-[m+1]x^m+1)\xi^m}{m!}$$

$$= \frac{xe^{x\xi}}{(x-1)^2}\left(x^2\xi - x\xi - 1 + e^{\xi(1-x)}\right) \tag{3.4.14}$$

Differentiating Δ with respect to ξ and η and adding gives

$$\xi\,\Delta_\xi(\xi,\eta) + \eta\theta\Delta\eta(\xi,\eta) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+n\theta)\,\Omega_{m,n}(x)\,\frac{\xi^m\eta^n}{m!n!}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m\,x\,\Omega_{m-1,n}(x)\,\frac{\xi^m\eta^n}{m!n!} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n\theta x\Omega_{m,n-1}(x)\,\frac{\xi^m\eta^n}{m!n!}$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m\,x\left(\frac{x^{m+n}-1}{x-1}\right)\frac{\xi^m\eta^n}{m!n!}\,. \qquad (3.4.15)$$

Taking the terms of the right hand side of (3.4.15) one at a time:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} m\,x\,\Omega_{m-1,n}(x)\,\frac{\xi^m\eta^n}{m!n!} = \xi x\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\Omega_{m-1,n}(x)\,\frac{\xi^{m-1}\,\eta!}{(m-1)!\,\eta!} = x\xi\Delta(\xi,\eta) \qquad (3.4.16)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n\theta x\,\Omega_{m,n-1}(x)\,\frac{\xi^m\eta^n}{m!n!} = \theta x\eta\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Omega_{m,n-1}(x)\,\frac{\xi^m\eta^{n-1}}{m!(n-1)!} = x\theta\eta\Delta(\xi,\eta) \qquad (3.4.17)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} mx\,\frac{x^{m+n}-1}{x-1}\,\frac{\xi^m\eta^n}{m!n!} = \frac{\xi x}{x-1}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(x^{m+n}-1)\frac{\xi^{m-1}\eta^n}{(m-1)!n!} = \frac{\xi x}{x-1}\left(xe^{x(\xi+\eta)}-e^{\xi+\eta}\right) \qquad (3.4.18)$$

Substituting (3.4.16) through (3.4.18) into (3.4.15) and simplifying we obtain

$$\xi(\Delta_\xi - x\Delta) + \theta\eta(\Delta_\eta - x\Delta) = \frac{\xi x}{x-1}\left(xe^{x(\xi+\eta)}-e^{\xi+\eta}\right). \qquad (3.4.19)$$

Let $\Delta(\xi,\eta) = e^{x(\xi+\eta)}\,V(\xi,\eta)$. Then (3.4.19), (3.4.13) and (3.4.14) become

$$\xi\,V_\xi(\xi,\eta) + \theta\eta V_\eta(\xi,\eta) = \frac{\xi x}{x-1}\left(x - e^{(1-x)(\xi+\eta)}\right) \qquad (3.4.20)$$

$$V(0,\eta) = 0 \qquad (3.4.21)$$

$$V(\xi,0) = \frac{x}{(x-1)^2}\left(x^2\xi - x\xi - 1 + e^{\xi(1-x)}\right). \qquad (3.4.22)$$

Let us apply the method of characteristics to solve (3.4.20):

$$\frac{d\xi}{\xi} = \frac{d\eta}{\eta} = \frac{dV}{\frac{\xi x}{x-1}\left(x-e^{(1-x)(\xi+\eta)}\right)} \qquad (3.4.23)$$

The first and second members yield $\eta = c_1\xi^{\theta}$.

The first and third members yield

$$\frac{dV}{d\xi} = \frac{x}{x-1}\left((x-e^{(1-x)(\xi+C_1\xi^{\theta})}\right) \quad \text{so that}$$

$$V(\xi,\eta) = \frac{x}{x-1}\int_0^{\xi} x-e^{(1-x)\tau+C_1\tau^{\theta})}d\tau.$$

Let $t = \frac{\tau}{\xi}$, $\xi dt = d\tau$. Then

$$V(\xi,\eta) = \frac{x\xi}{x-1}\int_0^1 x - e^{(1-x)(\xi t + \eta t^{\theta})}dt. \qquad (3.4.24)$$

We see that the boundary conditions (3.4.21) and (3.4.22) are satisfied since $V(0,\rho) = 0$ and

$$V(\xi,0) = \frac{x\xi}{x-1}\left(x - \frac{1}{\xi(1-x)}\left[e^{(1-x)\xi}-1\right]\right)$$

$$= \frac{x}{(x-1)^2}\left(x^2\xi - x\xi + e^{(1-x)\xi}-1\right)$$

Therefore

$$\Delta(\xi,\eta) = \frac{x\xi e^{x(\xi+\eta)}}{x-1}\left(x - \int_0^1 e^{(1-x)(\eta t + \eta t^{\theta})}dt\right)$$

$$= \frac{x}{x-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(x^{m+n+1}-\int_0^1 [(1-x)t+x]^m[(1-x)t^{\theta}]^n dt\right)\frac{\xi^{m+1}\eta^n}{m!n!}$$

(3.4.25)

Which yields $\Omega_{m,n}(x)$ as the coefficient of $\frac{\xi^m \eta^n}{m!n!}$:

$$\Omega_{m,n}(x) = \frac{mx}{x-1}\left(x^{m+n} - \int_0^1 [(1-x)t+x]^{m-1}[(1-x)t^\theta+x]^n dt\right). \quad (3.4.26)$$

To recover $D_\theta(m + n - r, m, n)$ we note

$$r!\, D_\theta(m+n-r,m,n) = \frac{d^r}{dx^r}\left(\Omega_{m,n}(x)\right)_{x=0}. \quad \text{From } (3.4.26)$$

$$\frac{d^r}{dx^r}\left(\Omega_{m,n}(x)\right)_{x=0} = m\, \frac{d^r}{dx^r}\left(\frac{x}{1-x}\int_0^1 [(1-t)x+t]^{m-1}[(1-t^\theta)x+t^\theta]^n dt\right)_{x=0}$$

$$= m \sum_{j=0}^{r} \binom{r}{j}\left[\frac{d^j}{dx^j}\int_0^1 [(1-t)x+t]^{m-1}[(1-t^\theta)x+t^\theta]^n dt\ \frac{d^{r-j}}{dx^{r-j}}\left(\frac{x}{1-x}\right)\right]_{x=0}$$

(3.4.27)

$$\text{Now}\quad \frac{d^{r-j}}{dx^{r-j}}\left(\frac{x}{1-x}\right)_{x=0} = \begin{cases} 0 & j = r \\ (r-j)! & j < r \end{cases} \quad (3.4.28)$$

and

$$\left[\frac{d^j}{dx^j}\int_0^1 [(1-t)x+t]^{m-1}[(1-t^\theta)x+t^\theta]^n dt\right]_{x=0}$$

$$= \sum_{k=0}^{j}\binom{j}{k}\int_0^1 \left[\frac{d^k}{dx^k}[(1-t)x+t]^{m-1}\frac{d^{j-k}}{dx^{j-k}}[(1-t^\theta)x+t^\theta]^n\right]_{x=0} dt$$

$$= \sum_{k=0}^{j}\binom{j}{k}(m-1)_k(n)_{j-k}\int_0^1 t^{m-1-k}(1-t)^k t^{\theta(n-j+k)}(1-t^\theta)^{k-j} dt$$

$$= \sum_{k=0}^{j} j!\binom{m-1}{k}\binom{n}{j-k}\int_0^1 (1-t)\,(1-t^\theta)^{k-j}\, t^{m-1-k+\theta(n-j+k)} dt$$

(3.4.29)

Substituting (3.4.28) and (3.4.29) into (3.4.27) gives:

$$E_\theta(A;m,n,r) = m \sum_{j=0}^{r-1} \sum_{k=0}^{j} \binom{m-1}{k}\binom{n}{j-k} \int_0^1 (1-t)^k (1-t^\theta)^{j-k}\, t^{m-k-1+\theta(n-j+k)}\, dt \tag{3.4.29}$$

which is certainly no improvement over (3.4.1).

Another approach is to recognize that

$$E_\theta(A;\ m,\ n,\ r) = \frac{1}{r!}\ \frac{\partial^r}{\partial x^r}\left[\left[\frac{\partial\Phi_{m,n}(x,y)}{\partial y}\right]_{y=1}\right]_{x=0}$$

where $\Phi_{m,n}$ is defined by (3.2.19). Using (3.2.43) we find

$$\frac{1}{r!}\left[\frac{\partial\Phi_{m,n}(x,y)}{\partial y}\right]_{y=1} = \frac{m(m-1)}{r!}\int_0^1 [(1-x)t + x]^{m-2}\,[(1-x)t^\theta + x]^n\, x(1-t)dt$$

$$+ \frac{mn\theta}{r!}\int_0^1 [(1-x)t + x]^{m-1}\,[(1-x)t^\theta + x]^{n-1}\, x(1-t)t^{\theta-1}dt \tag{3.4.30}$$

Denote the first term on the right hand side of (3.4.30) by $h(x)$ and apply Leibniz's rule:

$$\left.\frac{\partial^r h(x)}{\partial x^r}\right|_{x=0} = \frac{m(m-1)}{r!}\int_0^1 \frac{\partial^r}{\partial x^r}\, x\left[(1-x)t + x\right]^{m-2}\left[(1-x)t^\theta + x\right]^n (1-t)dt$$

$$= \frac{m(m-1)}{r!}\int_0^1 \sum_{k=0}^{r}\binom{r}{k}\frac{\partial^k}{\partial x^k}\, x\ \frac{\partial^{r-k}}{\partial x^{r-k}}\left[(1-x)t+x\right]^{m-2}\left[(1-x)t^\theta+x\right]^n (1-t)dt\Bigg|_{x=0}$$

$$= \frac{m(m-1)}{r!}\int_0^1 r\ \frac{\partial^{r-1}}{\partial x^{r-1}}\left[(1-x)t + x\right]^{m-2}\left[(1-x)t^\theta + x\right]^n (1-t)dt\Bigg|_{x=0}$$

$$= \frac{m(m-1)}{(r-1)!} \int_0^1 \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{\partial^k}{\partial x^k}\Big[(1-t)x + t\Big]^{m-2} \frac{\partial^{r-1-k}}{\partial x^{r-1-k}}\Big[(1-t^{\theta})x + t^{\theta}\Big]^n (1-t)\,dt\Big|_{x=0}$$

$$= \frac{m(m-1)}{(r-1)!} \sum_{k=0}^{r-1}\binom{r-1}{k} \int_0^1 (m-2)_k t^{m-2-k}(1-t)^{k+1}(n)_{r-1-k} t^{\theta(n-r+1+k)}(1-t^{\theta})^{r-1-k}dt$$

$$= m(m-1) \sum_{k=0}^{r-1} \binom{m-2}{k}\binom{n}{r-k-1} \int_0^1 (1-t)^{k+1} (1-t^{\theta})^{r-k-1} t^{m-2-k+\theta(n-r+k+1)} dt. \tag{3.4.31}$$

Similarly, the second term on the right hand side of (3.4.30), when differentiated r times and x placed equal to zero gives

$$mn\theta \sum_{k=0}^{r-1} \binom{m-1}{k}\binom{n-1}{r-k-1} \int_0^1 (1-t)^{k+1} (1-t^{\theta})^{r-k-1} t^{m-2-k+\theta(n-r+k+1)} dt. \tag{3.4.32}$$

Combining (3.4.31) and (3.4.32) gives

$$E_{\theta}\{A;m,n,r\} = m \sum_{k=0}^{r-1} \Big\{(m-1)\binom{m-z}{k}\binom{n}{r-k-1} + n\theta\binom{m-1}{k}\binom{n-1}{r-k-1}\Big\} \int_0^1 (1-t)^{k+1} (1-t^{\theta})^{r-k-1} t^{m-2-k+\theta(n-r+k+1)} dt. \tag{3.4.33}$$

Writing $(1-t)^{k+1} = (1-t)^k - t(1-t)^k$ and breaking up the integral in (3.4.33) into two parts we obtain

$$E_{\theta}\{A;\, m,\, n,\, r\}$$
$$= \sum_{k=0}^{r-1} k\Big\{m\binom{m-1}{k}\binom{n}{r-1-k} + n\theta\binom{m}{k}\binom{n-1}{r-1-k}\Big\} \int_0^1 (1-t)^k (1-t^{\theta})^{r-1-k} t^{m-k-1+\theta(n-r+1-k)} dt$$

$$+ \sum_{k=0}^{r-1}\binom{m-1}{k}\binom{n}{r-1-k} \int_0^1 (1-t)^k (1-t^{\theta})^{r-1-k} t^{m-k-1+\theta(n-r+1-k)} dt$$

$$= E_\theta\{A;\ m,\ n,\ r-1\}$$

$$+\ m \sum_{k=0}^{r-1} \binom{m-1}{k}\binom{n}{r-1-k} \int_0^1 (1-t)^k (1-t^\theta)^{r-1-k} t^{m-k-1+\theta(n-r+1-k)} dt. \quad (3.4.34)$$

But, since $A = \sum_{j=1}^{r} X\ (j)$, we have

$E_\theta\{A;\ m,\ n,\ r\} = E_\theta\{A;\ m,\ n,\ r-1\} + P\{X(r) = 1\}$ and hence, in (3.4.34), we can identify the unconditional probability of drawing an element from M on the r^{th} draw as

$$P\{X(r)=1\} = m \sum_{k=0}^{r-1} \binom{m-1}{k}\binom{n}{r-1-k} \int_0^1 (1-t)^k (1-t^\theta)^{r-1-k} t^{m-k-1+\theta(n-r+1-k)} dt \quad (3.4.35)$$

which is a valuable result itself.

Equation (3.4.35) gives us insight into interpreting (3.4.29) Replacing j by $j-1$ in (3.4.29) gives

$$E_\theta(A;m,n,r) = m \sum_{j=1}^{r} \sum_{k=0}^{j-1} \binom{m-1}{k}\binom{n}{j-1-k} \int_0^1 (1-t)^k (1-t^\theta)^{j-1-k} t^{m-k-1+\theta(m-j+1+k)} dt$$

$$= \sum_{j=1}^{r} P\{X(j) = 1\}$$

so that (3.4.29) simply states

$$E_\theta(A;\ m,\ n,\ r) = \sum_{j=1}^{r} E\ [X(j)].$$

SECTION 4

ASYMPTOTIC RESULTS

4.1 When m and n are large compared to r.

Using (3.1.4) we shall obtain crude bounds on $P_\theta(A; m, n, r)$ which converge to the same quantity as m and n go to infinity while $m/n = \eta$ is held fixed. We first prove a lemma concerning each term on the right hand side of (3.1.4).

Lemma 4.1.1 For all k and j such that

$$1 \leq k \leq r \quad \text{and} \quad 1 \leq j \leq \binom{r}{a},\ \theta \begin{Bmatrix} > \\ < \end{Bmatrix} 1 \quad \text{implies}$$

$$m(k, x_{\binom{r}{a}}) + \theta n(k, x_{\binom{r}{a}}) \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} m(k,x_j) + \theta n(k,x_j) \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} m(k,x_1) + \theta n(k,x_1).$$

Proof: Let $\theta = 1 + \epsilon$, $\epsilon > -1$. For all k and j such that $1 \leq k \leq r$, $1 \leq j \leq \binom{r}{a}$

$$\begin{aligned} m(k,x_j) + \theta n(k,x_j) &= m(k,x_j) + n(k,x_j) + \epsilon n(k,x_j) \\ &= m+n-k+1 + \epsilon n(k,x_j) \quad \text{by} \quad (3.1.2). \end{aligned}$$

By the definition of $n(k,x_j)$ and the ordering imposed on $X(a)$ we have

$$n(k, x_{\binom{r}{a}}) \leq n(k,x_j) \leq n(k,x_1).$$

Multiplying this inequality through by ϵ (the inequalities are reversed if $\epsilon < 0$) and adding $m+n-k+1$ to each term proves the lemma.

$$\text{Let} \quad \mu = m + n\theta \quad \text{and} \quad (d)_m = d(d-1)\ldots(d-m+1).$$

Using (3.1.2) and the definition of x_1 we see that

$$\prod_{k=1}^{r}\left[m(k,x_1)+\theta n(k,x_1)\right]^{-1}=\left[\mu(\mu-1)\ldots(\mu-a)(\mu-a-\theta)\ldots(\mu-a-[r-a-1]\theta)\right]^{-1}$$

$$=\left[(\mu)_a\left(\frac{\mu-a}{\theta}\right)_{r-a}\theta^{r-a}\right]^{-1} \qquad (4.1.1)$$

and, using the definition of $x_{\binom{r}{a}}$

$$\prod_{k=1}^{r}\left[m(k,x_{\binom{r}{a}})+\theta n(k,x_{\binom{r}{a}})\right]^{-1}=\left[\mu(\mu-\theta)\ldots(\mu-[r-a]\theta)(\mu-[r-a]\theta-1)\ldots\right.$$

$$\left.(\mu-[r-a]\theta-a+1)\right]^{-1}$$

$$=\left[\left(\frac{\mu}{\theta}\right)_{r-a}(\mu-[r-a]\theta)_a\,\theta^{r-a}\right]^{-1}. \qquad (4.1.2)$$

Using lemma 4.1.1 and substituting (4.1.1) and (4.1.2) into (3.1.4) for $\theta \begin{Bmatrix} > \\ < \end{Bmatrix} 1$ gives

$$\binom{r}{a}\frac{(m)_a\,(n)_{r-a}}{(m+n\theta)_a\left(n+\frac{m-a}{\theta}\right)_{r-a}}\begin{Bmatrix}\le\\ \ge\end{Bmatrix}P_\theta\{a;m,n,r\}\begin{Bmatrix}\le\\ \ge\end{Bmatrix}\binom{r}{a}\frac{(m)_a\,(n)_{r-a}}{(m+[n-r+a]\theta)_a\left(n+\frac{m}{\theta}\right)_{r-a}}. \qquad (4.1.3)$$

For $\theta = 1$, the bounds coincide and equal the correct central hypergeometric probability.

<u>Lemma 4.1.2</u>

$\lim_{m\to\infty}\frac{(am+b)_\alpha}{(cm+d)_\alpha}=\left(\frac{a}{c}\right)^\alpha$ where a,b and d are real, α is any positive integer and $c \neq 0$.

Proof: The proof is immediate by writing:

$$\lim_{m\to\infty} \frac{(am+b)_\alpha}{(cm+d)_\alpha} = \lim_{m\to\infty} \frac{(a+\frac{b}{m})(a+\frac{b-1}{m})(a+\frac{b-2}{m})\ldots(a+\frac{b-\alpha+1}{m})}{(c+\frac{d}{m})(c+\frac{d-1}{m})(c+\frac{d-2}{m})\ldots(c+\frac{d-\alpha+1}{m})} = (\frac{a}{c})^\alpha .$$

Let $\eta = \frac{m}{n}$ in (4.1.3) and apply lemma 4.1.2 to both the upper bound U and the lower bound L:

$$U = \binom{r}{a} \frac{(m)_a (\eta m)_{r-a}}{([1+\eta\theta]m + [a-r]\theta)_a([\eta+\frac{1}{\theta}]m)_{r-a}} \longrightarrow \binom{r}{a} (\frac{1}{1+\eta\theta})^a (\frac{\eta\theta}{1+\eta\theta})^{r-a} \qquad (4.1.4)$$

$$L = \binom{r}{a} \frac{(m)_a (\eta m)_{r-a}}{([1+\eta\theta]m)_a([\frac{1}{\theta}+\eta]m - \frac{a}{\theta})_{r-a}} \longrightarrow \binom{r}{a} (\frac{1}{1+\eta\theta})^a (\frac{\eta\theta}{1+\eta\theta})^{r-a} \qquad (4.1.5)$$

so that the bounds, and hence $P_\theta(a;\, m,\, n,\, r)$, both approach a binomial probability with parameters r and $p = \frac{1}{1+\eta\theta}$. Therefore, if r is small with respect to m and n, one can use the properties of the binomial distribution to investigate pertinent questions such as estimation and moment problems. For example, the usual minimum variance unbiased estimate of p in a binomial distribution can be used to estimate θ by

$$\frac{a}{r} = \hat{p} = \frac{1}{1+\eta\hat{\theta}} \quad \text{implies} \quad \hat{\theta} = \frac{m}{n} \Big/ \frac{a}{(r-a)} \qquad (4.1.6)$$

and the first two moments of the distribution $P_\theta(A;\, m,\, n,\, r)$ are

$$\mu = \frac{rm}{m+n\theta} \qquad (4.1.7)$$

$$\sigma^2 = \frac{r\, m\, n\, \theta}{(m+n\theta)^2} .$$

For $m = n = 20$, table 4.1 shows the comparative values of the exact probabilities $P_\theta\{a;\ 20,\ 20,\ r\}$ with the binomial approximation $B(p,\ r)$ with $p = \frac{1}{1+\theta}$.

Table 4.1

r	a	$P_\theta\{a;\ 20,20,r\}$		$B(\frac{1}{1+\eta\theta},\ r)$	
		$\theta = 2$	$\theta = 4$	$\theta = 2$	$\theta = 4$
3	0	.2808	.4951	.2963	.5120
	1	.4599	.4003	.4444	.3840
	2	.2260	.0970	.2222	.0960
	3	.0333	.0070	.0370	.0080
5	0	.1088	.2918	.1317	.3277
	1	.3278	.4268	.3292	.4096
	2	.3534	.2234	.3292	.2048
	3	.1704	.0523	.1646	.0512
	4	.0368	.0055	.0412	.0064
	5	.0028	.0002	.0041	.0003

4.2. When m, n, and r are large.

Let us now investigate the behavior of $P_\theta\{A;\ m,\ n,\ r\}$ as given in (3.2.46) as m, n and r approach infinity while the ratios $\frac{n}{m} = \eta$ and $\frac{r}{m} = \rho$ are held fixed.

Lemma 4.2.1 (Laplace's method)

Let $h(t)$ be a twice continuously differentiable real function of the real variable t on $\alpha < t < \beta$ and let $g(t)$ be continuous on $\alpha < t < \beta$. Let $h(t)$ have a unique maximum at $t = t_o$, $\alpha < t_o < \beta$, and $g(t_o) \neq 0$. Then, for large m,

$$\int_\alpha^\beta g(t)e^{mh(t)}dt \sim \sqrt{\frac{2\pi}{-mh''(t_o)}}\ g(t_o)e^{mh(t_o)} \tag{4.2.1}$$

Remark: ① If $h(t)$ has a finite number of maxima, the integral can be broken into a finite sum of integrals, each satisfying the unique maximum requirement.

② If $t_o = \beta$ and $h(t)$ is well behaved at $t = \beta$, then the right hand side of (4.2.1) must be halved.

Lemma 4.2.2

Let $\frac{r}{m} = \rho > 0$, $\frac{n}{m} = \eta > 0$ and $t_o = \left[\frac{1+\theta(\eta-\rho)}{1+\theta\eta}\right]^{\frac{1}{\theta}}$. Then, for large m,

$$\int_0^1 (1-t)^a (1-t^\theta)^{r-a} t^{m-a-1+\theta(n-r+a)} dt$$

$$\sim \frac{1}{\theta}\sqrt{\frac{2\pi}{r}}\ (1-t_o)^a (1-t_o^\theta)^{r-a+1}\ t_o^{m-a+\theta(n-r+a-\frac{1}{2})} .$$

Proof: Assume the parameters have been labeled so that $r \leq n$. Thus can always be done without loss of generality and insures $0 < t_o < 1$. Rewriting the integral in the form

$$\int_0^1 g(t)e^{mh(t)} dt, \quad \text{where } g(t) = \left(\frac{1-t}{1-t^\theta}\right)^a t^{a(\theta-1)-1}$$

and $h(t) = \rho \ell n (1-t^\theta) + [1+\theta(\eta-\rho)]\ell n\, t$, we note that $h(t)$ is twice continuously differentiable on $(0,1)$ and $g(t)$ is continuous on $(0,1)$. Furthermore, $h'(t) = 0$ has the unique solution

$t_o = \left[\frac{1+\theta(\eta-\rho)}{1+\theta\eta}\right]^{\frac{1}{\theta}}$ and $h''(t_o) = -\dfrac{\rho\theta^2 t_o^\theta}{t_o^2(1+t_o^\theta)^2} < 0$ so that t_o is the unique maximum of $h(t)$ on $(0,1)$. Lemma 4.2.1 therefore applies and, after some simplification, the stated result is obtained.

Lemma 4.2.3 For $\frac{r}{m} = \rho > 0$, $\frac{n}{m} = \eta > 0$

(i) $\binom{m}{a} \sim \frac{1}{a!}\, m^{\frac{1}{2}}\, (m-a)^{a-\frac{1}{2}}$

(ii) $\binom{n}{r-a} \sim \dfrac{n^{n+\frac{1}{2}}}{\sqrt{2\pi}\,(m-r+a)^{n-r+a+\frac{1}{2}}\,(r-a)^{n-a+\frac{1}{2}}}$

Proof: (i) By Stirling's formula

$$\binom{m}{a} = \frac{1}{a!}\,\frac{m!}{(m-a)!} \sim \frac{1}{a!}\,\frac{\sqrt{2\pi m}\,\left(\frac{m}{e}\right)^{m}}{\sqrt{2\pi(m-a)}\,\left(\frac{m-a}{e}\right)^{m-a}}$$

$$= \frac{1}{a!}\sqrt{\frac{m}{m-a}}\; e^{-a}\left(1-\frac{a}{m}\right)^{-m}(m-a)^{a}$$

$$\sim \frac{1}{a!}\, m^{\frac{1}{2}}\,(m-a)^{a-\frac{1}{2}}$$

(ii) $$\binom{n}{r-a} = \frac{\eta m!}{(\rho m-a)!\,([\eta-\rho]m+a)!}$$

$$\sim \frac{\sqrt{2\pi\eta m}\,\left(\frac{\eta m}{e}\right)^{\eta m}}{\sqrt{2\pi(\rho m-a)}\,\left(\frac{\rho m-a}{e}\right)^{\rho m-a}\sqrt{2\pi([\eta-\rho]m+a)}\,\left(\frac{[\eta-\rho]m+a}{e}\right)^{[\eta-\rho]m+a}}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(r-a)(n-r+a)}}\left(\frac{\eta m}{(\eta-\rho)m+a}\right)^{nm}\left(\frac{(\eta-\rho)m+a}{\rho m-a}\right)^{\rho m}\left(\frac{\rho m-a}{[\eta-\rho)m+a}\right)^{a}$$

$$\sim \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n}{(r-a)(n-r+a)}}\left(\frac{n}{n-r+a}\right)^{n}\left(\frac{n-r+a}{r-a}\right)^{r}\left(\frac{r-a}{n-r+a}\right)^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{n^{n+\frac{1}{2}}}{(n-r+a)^{n-r+a+\frac{1}{2}}\,(r-a)^{r-a+\frac{1}{2}}}$$

Now, applying lemmas 4.2.2 and 4.2.3, we obtain the asymptotic expansion

$$P_\theta\{a;\ m,\ n,\ r\} \sim \frac{1}{a!}\left(\frac{m-a+\theta[n-r+a]}{\theta[n-r+a]}\right)\left(\frac{mn\ t_o^\theta}{r(r-a)(m-a)(n-r+a)}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{(m-a)(r-a)(1-t_o)t_o^\theta}{(n-r+a)(1-t_o^\theta)t_o}\right)^a\left(\frac{(n-r+a)(1-t_o^\theta)}{(r-a)t_o^\theta}\right)^{r+1}\left(\frac{n\ t_o^\theta}{n-r+a}\right)^n t_o^m\ . \tag{4.2.2}$$

4.3. <u>When m, n, r and a are large</u>:

Using lemma 4.2.1 we now prove

<u>Lemma 4.3.1</u>.

Let $\frac{r}{m} = \rho > o$, $\frac{n}{m} = \eta > o$, $\frac{a}{m} = \alpha > o$. Then there exists a unique solution $t_1 \in (0,1)$ of the equation

$$(1+\eta)t^{\theta+1} - (1+\theta\eta-\alpha)t^\theta - (1+\theta[\eta-\rho+\alpha])t+1-\alpha+\theta[\eta-\rho+\alpha] = 0$$

and for large m,

$$\int_0^1 (1-t)^a(1-t^\theta)^{r-a}t^{m-a-1+\theta(n-r+a)}dt$$

$$\sim\left(\frac{2\pi}{at_1(1-t^\theta)^2+(r-a)\theta^2t_1^\theta(1-t_1)^2}\right)^{\frac{1}{2}}(1-t_1)^{a+1}(1-t_1^\theta)^{r-a-1}t_1^{m-a+\theta(n-r+a)}\ . \tag{4.3.1}$$

Proof: Rewriting the integral in the form

$$\int_0^1 g(t)e^{mh(t)}dt \quad\text{where}\quad g(t) = t^{-1} \quad\text{and}\quad h(t) = \ell n(1-t)^\alpha(1-t^\theta)^{\rho-\alpha}t^{1-\alpha+\theta[\eta-\rho+\alpha]},$$

we note that $g(t)$ is continuous in $(0,1)$ and $h(t)$ is twice continuosly differentiable. Furthermore, $h(o) = h(1) = -\infty$ so that there exists at least one relative maximum of $h(t)$ in $(0,1)$. Let t_1 be any solution in $(0,1)$ of

$$h'(t) = -\frac{\alpha}{1-t} - \frac{(\rho-\alpha)\theta t^{\theta-1}}{1-t^{\theta}} + \frac{1-\alpha+\theta[\eta-\rho+\alpha]}{t} = 0.$$

Simplifying, let t_1 be any solution of

$$(1+\theta\eta)t^{\theta+1} - (1+\theta\eta-\alpha)t^{\theta} - (1+\theta[\eta-\rho+\alpha])t+1-\alpha+\theta[\eta-\rho+\alpha] = 0, \quad (4.3.2)$$

in $(0,1)$. To show that there is only one such solution, it suffices to show that $h''(t_1) < 0$ since a continuously differentiable function cannot have two relative maxima without a relative minimum separating them.

$$h''(t) = -\frac{\alpha}{(1-t)^2} - \frac{(\rho-\alpha)\theta t^{\theta-2}(\theta-1+t^{\theta})}{(1-t^{\theta})^2} - \frac{1-\alpha+\theta(\eta-\rho+\alpha)}{t^2}$$

$$h''(t_1) = -\frac{\alpha}{(1-t_1)^2} - \frac{(\rho-\alpha)\theta^2 t_1^{\theta-2}}{(1-t_1^{\theta})^2} + \left[\frac{(\rho-\alpha)\theta t_1^{\theta-1}}{1-t_1^{\theta}} - \frac{1-\alpha+\theta(\eta-\rho+\alpha)}{t_1}\right]\frac{1}{t_1}$$

$$= -\frac{\alpha}{(1-t_1)^2} - \frac{(\rho-\alpha)\theta^2 t_1^{\theta-2}}{(1-t_1^{\theta})^2} + \frac{\alpha}{(1-t_1)t_1}$$

$$= -\frac{\alpha}{(1-t_1)^2 t_1} - \frac{(\rho-\alpha)\theta^2 t_1^{\theta}}{(1-t_1^{\theta})^2 t_1^2} < 0 \quad \text{since} \quad \rho - \alpha \geq 0$$

and $0 < t_1 < 1$. Hence, t_1 is the unique maximum of $h(t)$. Applying lemma 4.2.1 then proves the lemma.

Using Stirling's formula we obtain

$$\binom{m}{a} \sim \sqrt{\frac{m}{2\pi a(m-a)}}\left(\frac{m}{m-a}\right)^m\left(\frac{m-a}{a}\right)^a \quad (4.3.3)$$

and $$\binom{n}{r-a} \sim \sqrt{\frac{n}{2\pi(r-a)(n-r+a)}}\left(\frac{n}{n-r+a}\right)^{n}\left(\frac{n-r+a}{r-a}\right)^{r-a}. \tag{4.3.4}$$

Then, using (4.3.1), (4.3.3) and (4.3.4) we obtain

$$P_\theta\{a;\ m,\ n,\ r\} \sim \left(\frac{mn(m-a+\theta[n-r+a])^2(1-t_1)^2(1-t_1^\theta)^2}{2\pi a(m-a)(r-a)(n-r+a)(at_1[1-t_1^\theta]^2+[r-a]\theta^2 t_1^\theta[1-t_1]^2)}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{(m-a)(1-t_1)}{at_1}\right)^{a}\left(\frac{(n-r+a)(1-t_1^\theta)}{(r-a)t_1^\theta}\right)^{r-a}\left(\frac{mt_1}{m-a}\right)^{m}\left(\frac{nt_1^\theta}{n-r+a}\right)^{n} \tag{4.3.5}$$

Tables 4.3.1 and 4.3.2 furnish examples of the accuracy of the asymptotic expansion (4.3.5) for moderate and large values of the parameters, respectively.

Table 4.3.1

$m = n = 20, \quad r = 14$

a	$\theta = 2$			$\theta = 3$		
	Exact	Approx.	$\triangle$	Exact	Approx.	$\triangle$
0	.00027	—	—	.00245	—	—
1	.00458	.00499	.00041	.02579	.02812	.00233
2	.03040	.03186	.00146	.10502	.10999	.00497
3	.10478	.10832	.00354	.22253	.22993	.00740
4	.21141	.21713	.00572	.27664	.28400	.00736
5	.26587	.27211	.00624	.21483	.21981	.00498
6	.21585	.22054	.00469	.10793	.11025	.00232
7	.11521	.11767	.00246	.03572	.03647	.00075
8	.04066	.04156	.00090	.00783	.00800	.00017
9	.00943	.00966	.00023	.00113	.00116	.00003
10	.00141	.00145	.00004	.00011	.00011	0
11	.00012	.00014	.00002	0	0	0
12	0	.00001	.00001	0	0	0
13	0	0	0	0	0	0
14	0	0	0	0	0	0

Table 4.3.2

m = 95, n = 120, r = 75

a	0 = .5			a	θ = 2		
	Exact	Approx.	Δ		Exact	Approx.	Δ
	0	0	0	0	0	0	0
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	0	0	0	10	0	0	0
30	.0001	.0001	0	11	.0001	.0001	0
31	.0002	.0002	0	12	.0003	.0003	0
32	.0005	.0005	0	13	.0008	.0008	0
33	.0012	.0012	0	14	.0021	.0021	0
34	.0028	.0028	0	15	.0049	.0049	0
35	.0059	.0059	0	16	.0105	.0105	0
36	.0014	.0014	0	17	.0199	.0200	.0001
37	.0202	.0203	.0001	18	.0341	.0343	.0002
38	.0332	.0333	.0001	19	.0530	.0533	.0003
39	.0501	.0503	.0002	20	.0747	.0751	.0004
40	.0698	.0700	.0002	21	.0958	.0963	.0005
41	.0894	.0898	.0004	22	.1119	.1124	.0005
42	.1056	.1060	.0004	23	.1193	.1198	.0005
43	.1147	.1152	.0005	24	.1162	.1168	.0006
44	.1146	.1150	.0004	25	.1037	.1042	.0005
45	.1052	.1056	.0004	26	.0848	.0852	.0004
46	.0887	.0891	.0004	27	.0637	.0640	.0003
47	.0686	.0689	.0003	28	.0439	.0441	.0002
48	.0486	.0489	.0003	29	.0278	.0279	.0001
49	.0316	.0317	.0001	30	.0162	.0163	.0001
50	.0187	.0188	.0001	31	.0087	.0087	0
51	.0102	.0102	0	32	.0043	.0043	0
52	.0050	.0050	0	33	.0020	.0020	0
53	.0023	.0023	0	34	.0008	.0008	0
54	.0009	.0009	0	35	.0003	.0003	0
55	.0003	.0003	0	36	.0001	.0001	0
56	.0001	.0001	0	37	0	0	0
57	0	0	0	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮				
75	0	0	0	75	0	0	0

SECTION 5

EXAMPLES

5.1 Example 1: A Genetics Problem

Let us analyze the data in table 2.2. Using the normal approximation to the central hypergeometric distribution with parameters $m = 95$, $n = 120$, and $r = 75$, Fisher's exact test requires rejection of the hypothesis of unbiased selection at the 5% (nominal) level if, and only if,

$$|a - \frac{rm}{m+n}| > K_{.025}\left(\frac{rmn(m+n-r)}{(m+n)^2(m+n-1)}\right)^{\frac{1}{2}}$$

where $K_{.025} = 1.96$ is the 97.5 percentile of the unit normal distribution. For the values appearing in table 2.2 we find $|a - \frac{rm}{m+n}| = 7.9$ and

$$K_{.025}\left(\frac{rmn(m+n-r)}{(m+n)^2(m+n-1)}\right)^{\frac{1}{2}} = 6.8$$

so that the null hypothesis is rejected. If we define

$f(\theta) = \sum_{a=27}^{40} p_\theta(a;\ 95,\ 120,\ 75)$, we can display the sensitivity of the above test in the form of an operating characteristic curve. For example, referring to Figure 5.1.1, we see the probability of accepting the hypothesis of randomness when indeed "nature" is "twice as hard" on the heterozygotic rabbits (i.e., $\theta = 2$) is 0.17.

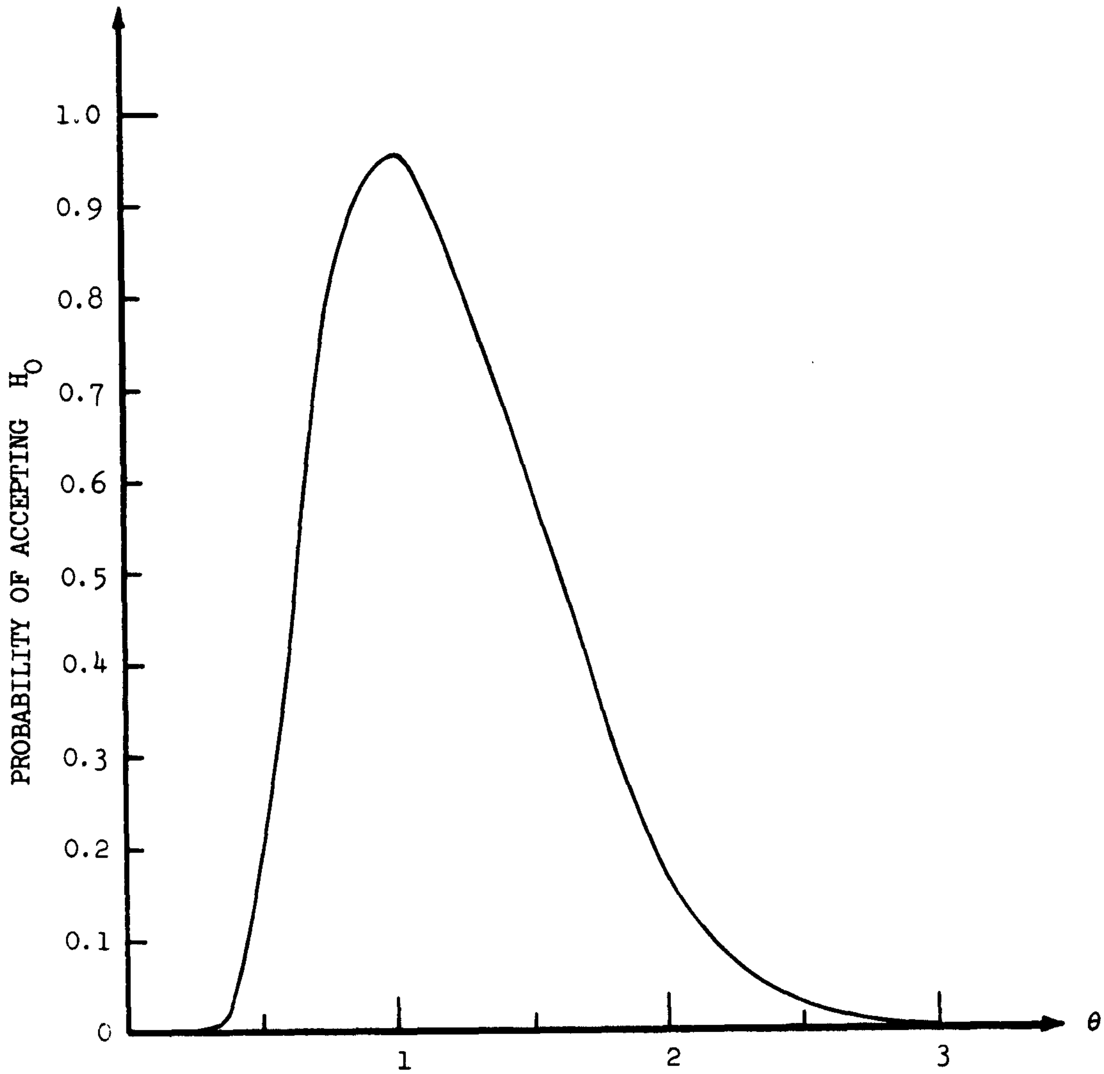

Figure 5.1.1.

5.2 <u>Example 2: A Hypothetical Problem and Associated Operating Characteristic Curves</u>.

Suppose a lot of transistors is to be divided among two purchasers. The lot is dichotomized by some quality criterion and let us assume there are m high quality items and n other items of tolerable quality. Purchaser G, who has contracted for r items, suspects the supplier of favoring purchaser H with respect to the distribution of the m high quality items. He therefore decides to test the hypothesis that the lot has been divided in a random manner (definition 1.1) against the one-sided alternative that the supplier is biased (in the sense of definition 2.1) in favor of purchaser H. Fisher's exact test is appropriate and the number of high quality items received by purchaser G has a noncentral hypergeometric distribution with parameters m, n, and r. In terms if the noncentrality parameter θ, purchaser G wishes to test

$$H_o : \quad \theta \leq 1$$

$$H_1 : \quad \theta > 1$$

Let A be the number of high quality items received by purchaser G in a shipment of r items. H_o is then rejected if, and only if, $A \leq A_o$ where A_o depends on m, n, r and the significance level α. Figures 5.2.1 - 5.2.3 show operating characteristics for $m = n = 10$ ($\alpha \approx .10$), $m = n = 12$ ($\alpha \approx .11$) and $m = n = 20$ ($\alpha \approx .05$). The curves are indexed by various values or r and the required A_o.

5.3 <u>The Noncentral Hypergeometric Distribution and a Class of Urn Problems</u>. In this subsection we shall consider a class of urn sampling problems which lead to the noncentral hypergeometric distribution. Suppose an urn contains m type 1 chips and n type 2 chips. A chip is drawn at random and kept with probability p_i if it is type i, $i = 1,2$. With probability $1 - p_i$ the drawn chip is returned to the urn and a second chip is then taken at random. This process is repeated until a chip is

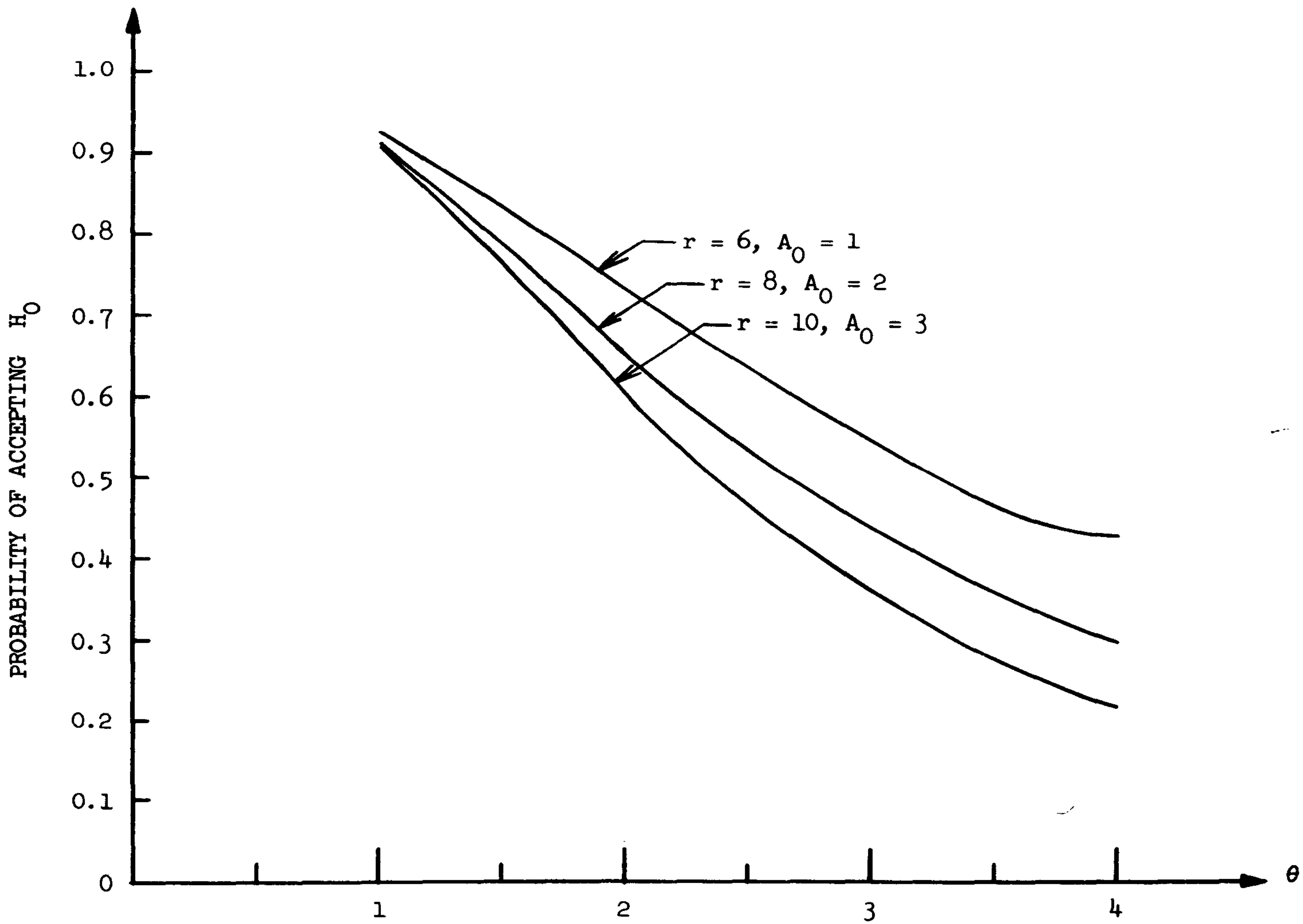

Figure 5.2.1. $m = n = 10$

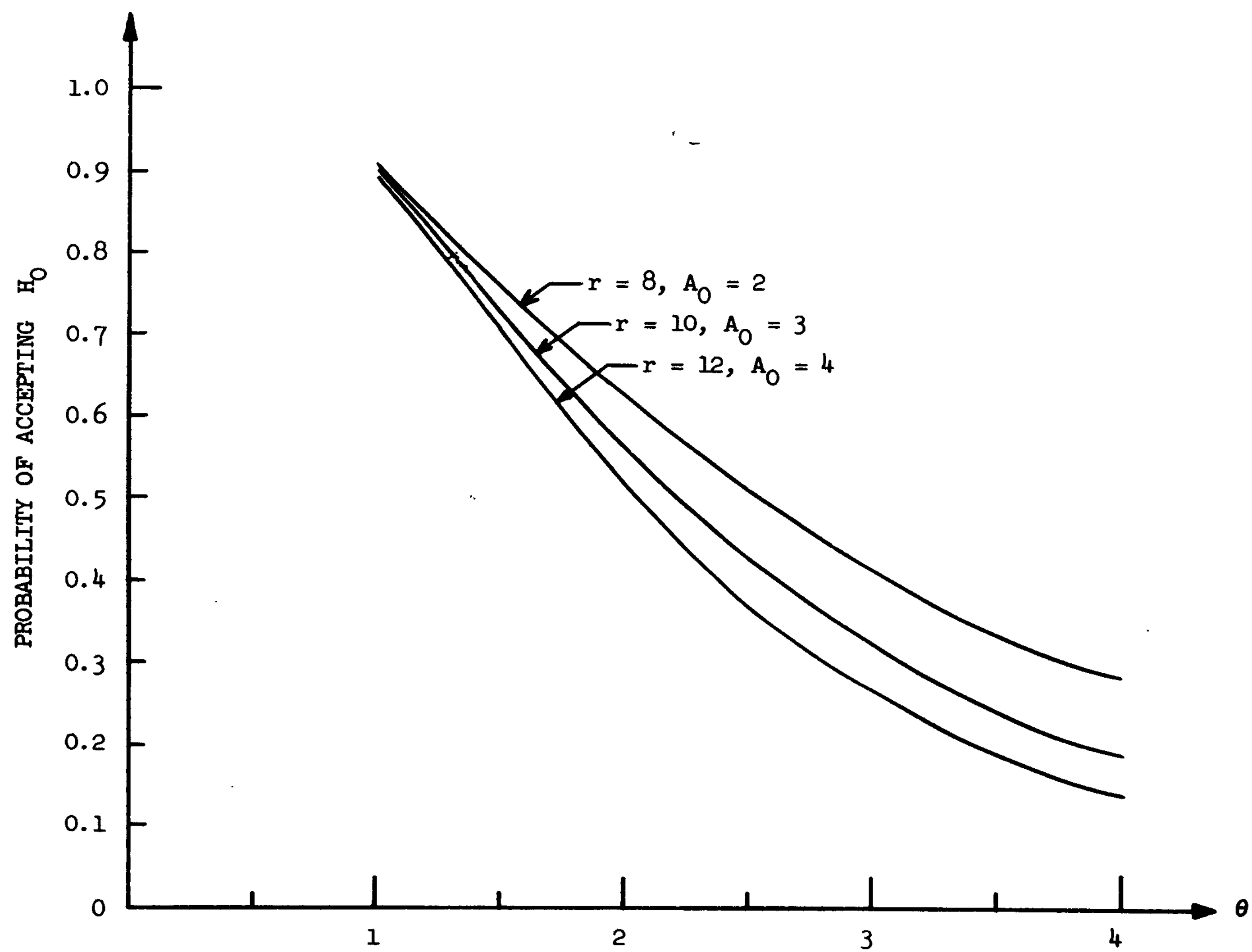

Figure 5.2.2. $m = n = 12$

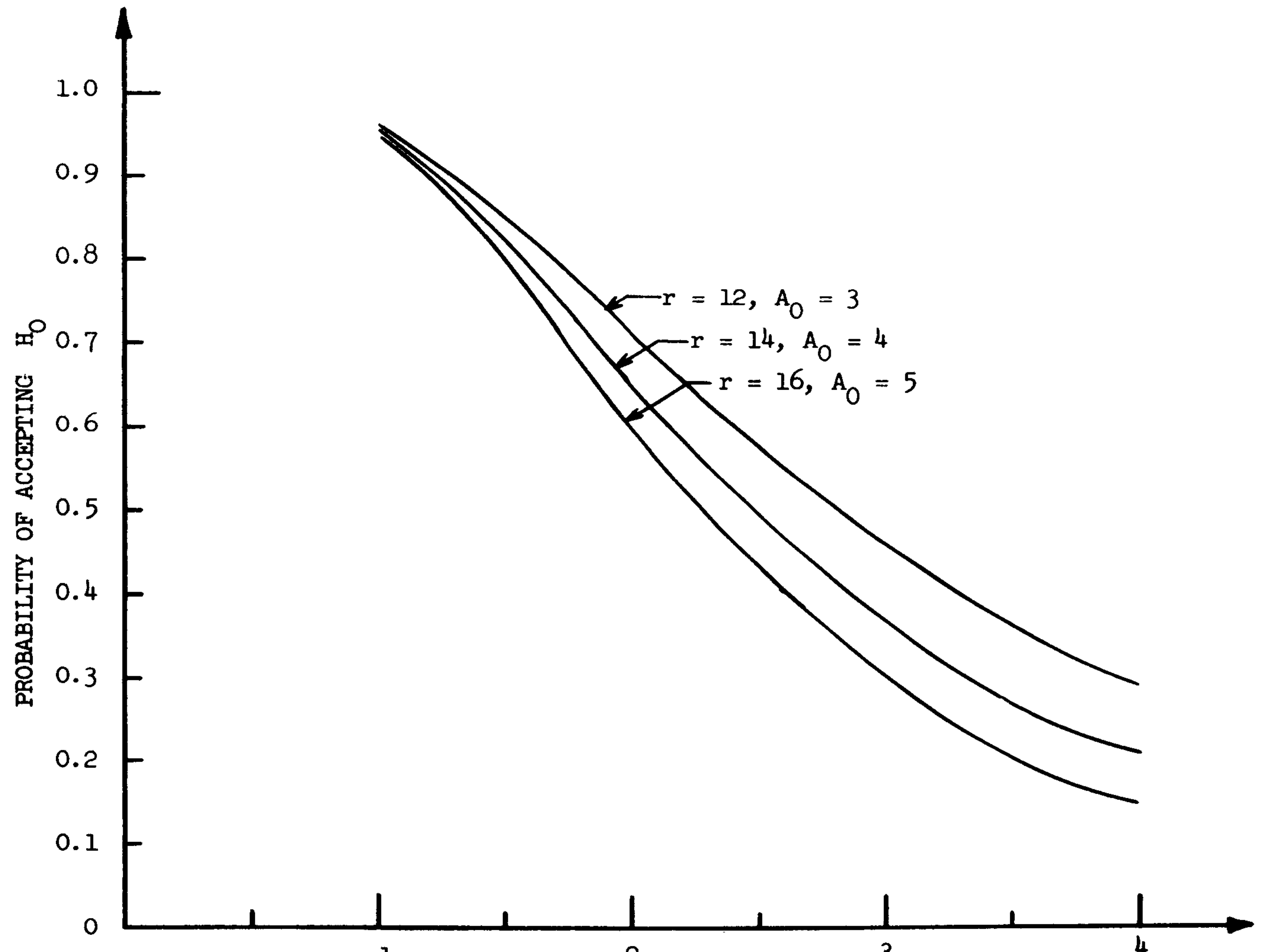

Figure 5.2.3. $m = n = 20$

retained. Let X and Y be random variables defined by

$X = i$ if the first drawn is a type i chip

$Y = i$ if the chip retained is a type i chip.

Clearly
$$P\{X = i\} = \begin{cases} \frac{m}{m+n} & i = 1 \\ \frac{n}{m+n} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

and $P\{Y=1\}$ is the probability that a type 1 chip is drawn and retained before a type 2 chip is drawn and retained. Hence,

$$\begin{aligned} P\{Y=1\} &= E[P\{Y=1|X\}] \\ &= P\{Y=1|X=1\}P\{X=1\} + P\{Y=1|X=2\}P\{X=2\} \\ &= [p_1 + (1-p_1)P\{Y=1\}] \frac{m}{m+n} + [(1-p_2)P\{Y=1\}] \frac{n}{m+n} \end{aligned}$$

Solving, we find $P\{Y=1\} = \dfrac{m}{m+(\frac{p_2}{p_1})n}$ and observe that the sampling mechanism has bias $\theta = \dfrac{p_2}{p_1}$ in the sense of definition 2.1. Therefore, if one continues sampling in manner described above until r chips have been retained, the number A of type 1 chips in the sample has a noncentral hypergeometric distribution with parameters m, n, and r and noncentrality depends only on the ratio of p_2 to p_1 so that the distribution of A is unchanged if, for example, $p_1 \geq p_2$ and type 1 chips are kept with probability 1 while type 2 chips are retained with probability $p^* = \dfrac{p_2}{p_1}$.

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